

Nonunique \mathcal{C} operator in \mathcal{PT} Quantum Mechanics

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(Dated: May 28, 2009)

Abstract

The three simultaneous algebraic equations, $\mathcal{C}^2 = 1$, $[\mathcal{C}, \mathcal{PT}] = 0$, $[\mathcal{C}, H] = 0$, which determine the \mathcal{C} operator for a non-Hermitian \mathcal{PT} -symmetric Hamiltonian H , are shown to have a nonunique solution. Specifically, the \mathcal{C} operator for the Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\mu^2 q^2 + i\epsilon q^3$ is determined perturbatively to first order in ϵ and it is demonstrated that the \mathcal{C} operator contains an infinite number of arbitrary parameters. For each different \mathcal{C} operator, the corresponding equivalent isospectral Dirac-Hermitian Hamiltonian h is calculated.

PACS numbers: 11.30.Er, 12.20.-m, 02.30.Mv, 11.10.Lm

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I. INTRODUCTION

A Hamiltonian H defines a physical theory of quantum mechanics if (i) H has a real energy spectrum, and (ii) the time-evolution operator $U = e^{-iHt}$ is unitary so that probability is conserved. These two features of the theory are guaranteed if H is Dirac Hermitian. (The Hamiltonian H is *Dirac Hermitian* if $H = H^\dagger$, where the symbol \dagger represents the combined operations of complex conjugation and matrix transposition.) However, it is not necessary for H to be Dirac Hermitian for the spectrum to be real and for time evolution to be unitary. For example, the Hamiltonians belonging to the class

$$H = p^2 + q^2(iq)^\epsilon \quad (\epsilon > 0) \quad (1)$$

possess real eigenvalues [1, 2] and generate unitary time evolution [3, 4]. These Hamiltonians therefore define physically acceptable quantum theories.

The Hamiltonians in (1) are \mathcal{PT} symmetric, that is, invariant under combined spatial reflection \mathcal{P} and time reversal \mathcal{T} . The underlying reason that the Hamiltonians H in (1) are physically acceptable is that they are selfadjoint, not with respect to the Dirac adjoint \dagger , but rather with respect to \mathcal{CPT} conjugation, where \mathcal{C} is a linear operator that represents a hidden reflection symmetry of H . The \mathcal{CPT} inner product defines a positive definite Hilbert space norm. Not every \mathcal{PT} -symmetric Hamiltonian has an entirely real spectrum but if the spectrum is entirely real, a linear \mathcal{PT} -symmetric operator \mathcal{C} exists that obeys the following three algebraic equations [3]:

$$\mathcal{C}^2 = 1, \quad (2)$$

$$[\mathcal{C}, \mathcal{PT}] = 0, \quad (3)$$

$$[\mathcal{C}, H] = 0. \quad (4)$$

When such a \mathcal{C} operator exists, we say that the \mathcal{PT} symmetry of H is *unbroken*. Constructing the \mathcal{C} operator is the key step in showing that time evolution for a non-Hermitian \mathcal{PT} -symmetric Hamiltonian (1) is unitary.

There has been much research activity during the past decade on non-Hermitian \mathcal{PT} -symmetric Hamiltonians [5] and the \mathcal{C} operator for some nontrivial quantum-mechanical models has been calculated perturbatively [6, 7, 8, 9]. The first perturbative calculation of \mathcal{C} was performed in Ref. [6] for the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\mu^2 q^2 + i\epsilon q^3. \quad (5)$$

Here, μ is a mass parameter, ϵ is a small coupling constant that is used as a perturbation parameter, and p and q are the usual quantum-mechanical operators satisfying the Heisenberg commutation relation $[q, p] = i$. It was shown in Ref. [6] that the \mathcal{C} operator has a simple and natural form as the parity operator \mathcal{P} multiplied by an exponential of a linear Dirac Hermitian operator \mathcal{Q} :

$$\mathcal{C} = e^{\mathcal{Q}}\mathcal{P}, \quad (6)$$

where $\mathcal{Q} = \mathcal{Q}^\dagger$. For the Hamiltonian (5), \mathcal{Q} is a series in *odd* powers of ϵ :

$$\mathcal{Q} = \mathcal{Q}_1\epsilon + \mathcal{Q}_3\epsilon^3 + \mathcal{Q}_5\epsilon^5 + \cdots. \quad (7)$$

To first order in ϵ the result for \mathcal{Q} was found to be

$$\mathcal{Q}_1 = -\frac{2}{\mu^2}qpq - \frac{4}{3\mu^4}p^3. \quad (8)$$

Note that in the unperturbed limit $\epsilon \rightarrow 0$ in which the Hamiltonian becomes Hermitian and parity symmetry is restored, the operator \mathcal{Q} vanishes and thus $\mathcal{C} \rightarrow \mathcal{P}$. This suggests that the \mathcal{C} operator may be interpreted as the complex extension of the parity operator \mathcal{P} .

Mostafazadeh showed that the \mathcal{Q} operator can be used to construct a similarity transform that maps the non-Dirac-Hermitian Hamiltonian H to a spectrally equivalent Dirac-Hermitian Hamiltonian h [10]:

$$h = e^{\mathcal{Q}/2} H e^{-\mathcal{Q}/2}. \quad (9)$$

This similarity transformation was originally used by Scholtz *et al.* to convert Hermitian Hamiltonians to non-Hermitian Hamiltonians [11].

In Sec. II of this paper we reinvestigate the Hamiltonian H in (5). We show that the perturbative solution in (8) for the \mathcal{Q} operator is in fact not unique and that the general solution to the algebraic system (2) – (4) contains an infinite number of arbitrary continuous parameters. Then, in Sec. III, we show that for each of these \mathcal{C} operators there is a corresponding spectrally equivalent Hermitian Hamiltonian h .

II. COMPLETE FIRST-ORDER CALCULATION OF THE \mathcal{C} OPERATOR

The purpose of this paper is to show that the \mathcal{PT} -symmetric Hamiltonian H in (5) has an infinite class of associated \mathcal{C} operators. To determine the \mathcal{C} operator for a given Hamiltonian H we must solve the three equations (2) – (4). The first two of these equations are kinematic constraints on \mathcal{C} ; they are obeyed by the \mathcal{C} operator for any \mathcal{PT} -symmetric Hamiltonian. If we seek a solution for \mathcal{C} in the form (6), we find that (2) and (3) imply that $\mathcal{Q}(p, q)$ is an *odd function of the p operator and an even function of the q operator*. The third equation (4) is dynamical because it makes explicit reference to the Hamiltonian. Our specific objective here is to solve (4) perturbatively to first order in ϵ .

We substitute (5) and (6) into (4) and to first order in ϵ obtain the commutation relation satisfied by \mathcal{Q}_1 :

$$[\mathcal{Q}_1, H_0] = 2iq^3, \quad (10)$$

where $H_0 = \frac{1}{2}p^2 + \frac{1}{2}\mu^2q^2$ is the unperturbed Hamiltonian. The simplest solution to (10) that is odd in p and even in q and is a *polynomial* in p and q is given by the solution in (8). Note that (8) is not the *unique* function of p and q that satisfies the commutation relation (10) because we can add to \mathcal{Q}_1 any function of H_0 . However, H_0 is an *even* function of p and \mathcal{Q}_1 is required to be an *odd* function of p . Thus, one is tempted to conclude (*wrongly!*) that the only solution to (10) is that given in (8).

To construct additional solutions to (10) we employ the operator notation described in detail in Ref. [12]. We introduce the set of totally symmetric operators $T_{m,n}$, where $T_{m,n}$ is an equally weighted average over all possible orderings of m factors of p and n factors of q . For example,

$$\begin{aligned} T_{0,0} &= 1, \\ T_{1,0} &= p, \\ T_{1,1} &= \frac{1}{2}(pq + qp), \\ T_{1,2} &= \frac{1}{3}(pqq + qpq + qqp), \\ T_{2,2} &= \frac{1}{6}(ppqq + qqpp + pqqp + qppq + qpqp + pqpq), \end{aligned} \quad (11)$$

and so on. Note that $T_{m,n}$ is the quantum-mechanical generalization of the classical product $p^m q^n$. We can express the polynomial solution \mathcal{Q}_1 in (8) to the commutation relation (10) as a linear combination of two such totally symmetric operators:

$$\mathcal{Q}_1 = -\frac{4}{3\mu^4}T_{3,0} - \frac{2}{\mu^2}T_{1,2}. \quad (12)$$

The advantage of the totally symmetric operators $T_{m,n}$ is that it is especially easy to evaluate commutators and anticommutators. For example, as shown in Ref. [12], the operators $T_{m,n}$ obey extremely simple commutation and anticommutation relations:

$$\begin{aligned} [p, T_{m,n}] &= -inT_{m,n-1}, \\ [q, T_{m,n}] &= imT_{m-1,n}, \\ \{p, T_{m,n}\} &= 2T_{m+1,n}, \\ \{q, T_{m,n}\} &= 2T_{m,n+1}. \end{aligned} \quad (13)$$

Also, by combining and iterating the results in (13), we can establish additional useful commutation relations for $T_{m,n}$. For example,

$$\begin{aligned} [p^2, T_{m,n}] &= -2inT_{m,n-1}, \\ [q^2, T_{m,n}] &= 2imT_{m-1,n}. \end{aligned} \quad (14)$$

The totally symmetric operators $T_{m,n}$ can be re-expressed in Weyl-ordered form [12]:

$$T_{m,n} = \frac{1}{2^m} \sum_{k=0}^m \binom{m}{k} p^k q^n p^{m-k} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} q^k p^m q^{n-k} \quad (m, n = 0, 1, 2, 3, \dots). \quad (15)$$

The proof that the totally symmetric form of $T_{m,n}$ in (11) equals the binomial-summation Weyl-ordered forms above requires the repeated use of the Heisenberg algebraic property that $[q, p] = i$ and follows by induction. The reason for introducing the Weyl-ordered form of $T_{m,n}$ is that it allows us to extend the totally symmetric operators $T_{m,n}$ to negative values of n by using the first of these formulas or to negative values of m by using the second of these formulas. The commutation and anticommutation relations in (13) and (14) remain valid if m is negative or if n is negative.

We will now show that (12) is just one of many solutions to (10) that are even in q and odd in p . Note that (10) is an *inhomogeneous* linear equation for \mathcal{Q}_1 and that (12) is a *particular* solution to this equation. To find other solutions, we need only solve the *associated homogeneous* linear equation

$$[\mathcal{Q}_{1, \text{homogeneous}}, H_0] = 0. \quad (16)$$

Let us look for a solution to this equation of the form

$$\mathcal{Q}_{1, \text{homogeneous}} = \sum_{k=0}^{\infty} a_k T_{3-2k, 2k}, \quad (17)$$

which by construction is odd in p and even in q . Substituting (17) into (16) and using the commutation relations in (14), we obtain a two-term recursion relation for the coefficients a_n :

$$a_{k+1} = -\frac{k - \frac{3}{2}}{k + 1} \mu^2 a_k \quad (k = 0, 1, 2, 3, \dots). \quad (18)$$

The solution to this recursion relation is

$$a_k = \frac{a}{\mu^4} \frac{\Gamma(k - \frac{3}{2})}{k! \Gamma(-\frac{3}{2})} (-\mu^2)^k \quad (k = 0, 1, 2, 3, \dots), \quad (19)$$

where a is an arbitrary constant. Thus, a one-parameter family of solutions to the homogeneous equation (16) is

$$\mathcal{Q}_{1, \text{homogeneous}} = \frac{a}{\mu^4} \sum_{k=0}^{\infty} \frac{\Gamma(k - \frac{3}{2})}{k! \Gamma(-\frac{3}{2})} (-\mu^2)^k T_{3-2k, 2k}. \quad (20)$$

We have included the factor of μ^4 in the denominator so that $\epsilon \mathcal{Q}_{1, \text{homogeneous}}$ is dimensionless.

Combining the inhomogeneous solution in (12) and the homogeneous solutions in (20) gives a general one-parameter class of solutions to the commutator condition (10):

$$\mathcal{Q}_1 = -\frac{4}{3\mu^4} T_{3,0} - \frac{2}{\mu^2} T_{1,2} + \frac{a}{\mu^4} \sum_{k=0}^{\infty} \frac{\Gamma(k - \frac{3}{2})}{k! \Gamma(-\frac{3}{2})} (-\mu^2)^k T_{3-2k, 2k}. \quad (21)$$

The coefficients in this series are indicated schematically by the diagonal line of a 's in Fig. 1 that intersects the m axis at $(3, 0)$ and the n axis at $(0, 3)$. On this figure we use circles to indicate the two terms in (21) that arise from the particular polynomial solution (12).

One might be concerned that because the operator p in the series in (21) is raised to arbitrarily high negative powers it might be difficult to interpret the series as an operator. However, we will now show how to sum this operator series to obtain a well defined and meaningful result. First, we observe that there is an extremely simple way to represent the totally symmetric operator $T_{-n,n}$ for all integers n :

$$T_{-n,n} = \frac{1}{2} \left(q \frac{1}{p} \right)^n + \frac{1}{2} \left(\frac{1}{p} q \right)^n. \quad (22)$$

This nontrivial formula can be verified by using the Heisenberg algebra $[q, p] = i$. One may regard this formula as the companion to the result that

$$T_{n,n} = \text{Hahn}_n(T_{1,1}), \quad (23)$$

where $\text{Hahn}_n(x)$ is a Hahn polynomial of degree n [13].

In order to use the formula (21), we replace the k th term in the series in (21) with a triple anticommutator of $T_{-2k, 2k}$:

$$T_{3-2k, 2k} = \frac{1}{8} \{ \{ \{ T_{-2k, 2k}, p \}, p \}, p \}. \quad (24)$$

We then recognize that the sum is just a pair of binomial expansions of the general form

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(k-\alpha)}{k! \Gamma(-\alpha)} (-x)^k \quad (25)$$

with $\alpha = \frac{3}{2}$, one for each of the two operators $q \frac{1}{p}$ and $\frac{1}{p} q$. Thus, the one-parameter family of solutions in (21) simplifies to

$$\begin{aligned} \mathcal{Q}_1 = & -\frac{4}{3\mu^4} T_{3,0} - \frac{2}{\mu^2} T_{1,2} + \frac{a}{16\mu^4} \left\{ \left\{ \left\{ \left(1 + \mu^2 q \frac{1}{p} q \frac{1}{p} \right)^{3/2}, p \right\}, p \right\}, p \right\} \\ & + \frac{a}{16\mu^4} \left\{ \left\{ \left\{ \left(1 + \mu^2 \frac{1}{p} q \frac{1}{p} q \right)^{3/2}, p \right\}, p \right\}, p \right\}. \end{aligned} \quad (26)$$

Note that at the classical level, where q and p commute, the p 's in the anticommutators combine with the expression to the power $\frac{3}{2}$ to give $H_0^{3/2}$, which is even in p . However, in (26) we can see that the symmetry requirement that Q be odd in p and even in q is satisfied! Furthermore, there are just enough powers of p in the triple anticommutators to cancel any small- p singularities.

One might now be tempted to argue as follows: In the Heisenberg algebra a commutator behaves as a derivative. Thus, the commutator equation (10) is analogous to a first-order linear ordinary differential equation. Since the solution to such an equation contains one arbitrary parameter, the solution in (26), which contains the arbitrary parameter a , should be the complete solution to (10). However, this argument is wrong and we will now show that there are in fact an *infinite number* of additional one-parameter families of solutions to (10).

We seek new one-parameter families of solutions to the associated homogeneous commutation relation (16), with each family of solutions labeled by the integer $P = 0, \pm 1, \pm 2, \pm 3, \dots$. For each value of P the solution has the form

$$\mathcal{Q}_{1, \text{homogeneous}}^{(P)} = \sum_{k=0}^{\infty} a_k^{(P)} T_{2P+3-2k, 2k}, \quad (27)$$

which is odd in p and even in q , as is required. We substitute (27) into (16) and use the

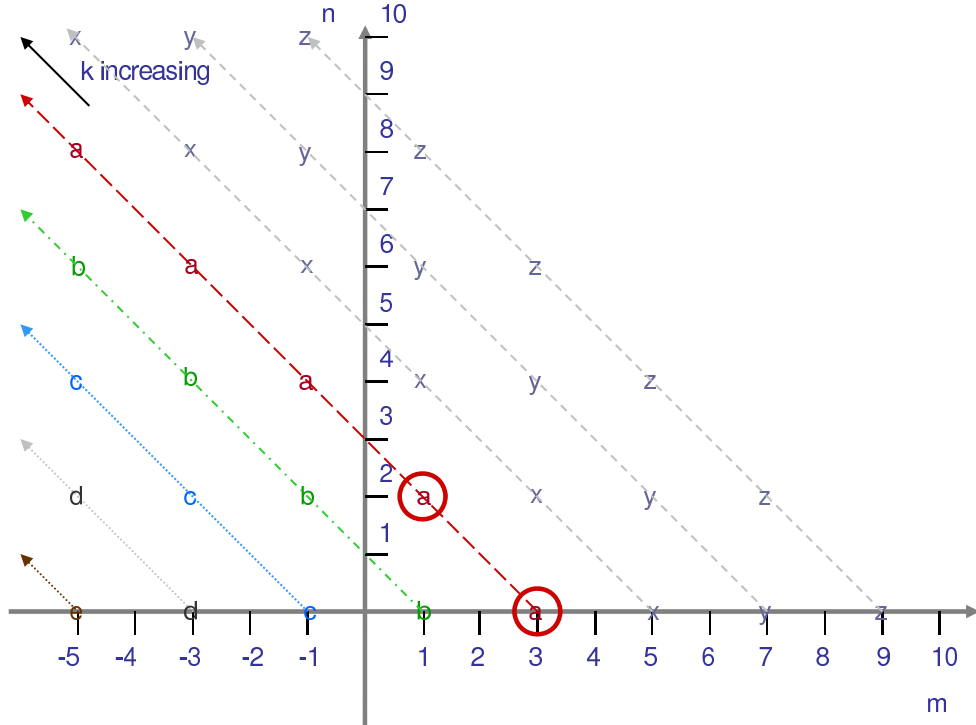


FIG. 1: Schematic representation of the coefficients in the operator series solution for Q_1 in (31). The two circles represent the terms in the inhomogeneous polynomial solution in (12). The a 's represent the coefficients in the operator series (20), which is the homogeneous solution corresponding to $P = 0$. The coefficients of the homogeneous solutions in (30) for $P = -1, -2$, and -3 are indicated by b 's, c 's, and d 's, and the coefficients in the homogeneous solutions for $P = 1, 2$, and 3 are indicated by x 's, y 's, and z 's.

commutation relations (14) to obtain a recursion relation for the coefficients $a_k^{(P)}$:

$$a_{k+1}^{(P)} = -\frac{k-P-\frac{3}{2}}{k+1}\mu^2 a_k^{(P)} \quad (k=0, 1, 2, 3, \dots). \quad (28)$$

The solution to this recursion relation is

$$a_k^{(P)} = \frac{a^{(P)}}{\mu^{P+4}} \frac{\Gamma(k-P-\frac{3}{2})}{k! \Gamma(-P-\frac{3}{2})} (-\mu^2)^k \quad (k=0, 1, 2, 3, \dots), \quad (29)$$

where $a^{(P)}$ are arbitrary constants. Thus, for each integer P we obtain the following one-parameter family of solutions to the homogeneous equation (16):

$$\mathcal{Q}_{1, \text{homogeneous}}^{(P)} = \frac{a^{(P)}}{\mu^{P+4}} \sum_{k=0}^{\infty} \frac{\Gamma(k-P-\frac{3}{2})}{k! \Gamma(-P-\frac{3}{2})} (-\mu^2)^k T_{2P+3-2k, 2k}. \quad (30)$$

The factor of μ^{P+4} in the denominator ensures that $\epsilon \mathcal{Q}_{1, \text{homogeneous}}^{(P)}$ is dimensionless. Note that if we set $P=0$, we recover the homogeneous solution in (20). In Fig. 1 the coefficients for the $P=-1$, $P=-2$, and $P=-3$ solutions are indicated by b 's, c 's, and d 's and the coefficients for the $P=1$, $P=2$, and $P=3$ solutions are indicated by x 's, y 's, and z 's.

We can now combine all these solutions to produce a very general solution for $\mathcal{Q}_1^{(P)}$:

$$\mathcal{Q}_1 = -\frac{4}{3\mu^4} T_{3,0} - \frac{2}{\mu^2} T_{1,2} + \sum_{P=-\infty}^{\infty} \frac{a^{(P)}}{\mu^{P+4}} \sum_{k=0}^{\infty} \frac{\Gamma(k-P-\frac{3}{2})}{k! \Gamma(-P-\frac{3}{2})} (-\mu^2)^k T_{2P+3-2k, 2k}. \quad (31)$$

This solution generalizes and replaces that in (21) and is one of the principal results in this paper. We emphasize that $a^{(P)}$ are all arbitrary.

Finally, we observe that it is possible to present the infinite sums over operators in (30) and (31) more compactly by performing the sum over k . To do so, we generalize the result in (24) as a $2P+3$ -fold anticommutator for the case when $P \geq -1$,

$$T_{2P+3-2k, 2k} = \frac{1}{2^{2P+3}} \{ \dots \{ T_{-2k, 2k}, p \}, \dots p \}, p \}_{2P+3 \text{ times}}. \quad (32)$$

This result allows us to apply the identity in (25) to perform the sum in (30). We find that

$$\begin{aligned} \mathcal{Q}_{1, \text{homogeneous}}^{(P)} &= \frac{a^{(P)}}{2^{2P+4} \mu^{P+4}} \left\{ \left\{ \dots \left\{ \left(1 + \mu^2 q \frac{1}{p} q \frac{1}{p} \right)^{P+3/2}, p \right\}, \dots p \right\}, p \right\}_{2P+3 \text{ times}} \\ &+ \frac{a^{(P)}}{2^{2P+4} \mu^{P+4}} \left\{ \left\{ \dots \left\{ \left(1 + \mu^2 \frac{1}{p} q \frac{1}{p} q \right)^{P+3/2}, p \right\}, \dots p \right\}, p \right\}_{2P+3 \text{ times}}. \end{aligned} \quad (33)$$

One may simplify this expression even further by exploiting the analog of the Baker-Hausdorff formula for multiple anticommutators [14]:

$$e^A B e^A = B + \{A, B\} + \frac{1}{2!} \{A, \{A, B\}\} + \frac{1}{3!} \{A, \{A, \{A, B\}\}\} + \dots. \quad (34)$$

Thus, the $2P+3$ -fold anticommutator in (32), for example, may be rewritten in a more compact form as a multiple derivative:

$$\{ \dots \{ T_{-2k, 2k}, p \}, \dots p \}, p \}_{2P+3 \text{ times}} = \frac{d^{2P+3}}{d\beta^{2P+3}} e^{\beta p} T_{-2k, 2k} e^{\beta p} \Big|_{\beta=0}. \quad (35)$$

When $P < -1$, we can use commutators instead of anticommutators to make the expressions more compact, but we do not present the results here because they are repetitive.

III. SPECTRALLY EQUIVALENT HAMILTONIANS

For the \mathcal{C} operators associated with Q_1 in (31) we must now calculate the Hermitian Hamiltonian h that is equivalent (in the sense that it is isospectral) to the Hamiltonian in H in (5). To do so, we evaluate the similarity transformation in (9), which to leading order in ϵ amounts to evaluating the commutator

$$h = H_0 + \frac{1}{4}i\epsilon^2 [q^3, Q_1]. \quad (36)$$

Note that when we use the *first-order* form of the Q operator, we obtain h to *second order* in ϵ . (The third-order contribution to Q gives h to fourth order in ϵ .)

In order to evaluate the commutator in (36) we use the relation

$$[q^3, T_{m,n}] = -\frac{1}{4}im(m-1)(m-2)T_{m-3,n} + 3imT_{m-1,n+2}, \quad (37)$$

which is derived from the commutation relations in (13) and (14). Furthermore, since (36) depends on Q_1 linearly, we can evaluate each of the contributions to Q_1 independently and add together the results at the end of the calculation.

First, we evaluate the commutator in (36) for the inhomogeneous part (the first two terms) in Q_1 and obtain

$$h = H_0 + \frac{\epsilon^2}{2\mu^4} (-1 + 6T_{2,2} + 3\mu^2 T_{0,4}). \quad (38)$$

This result was already obtained in Ref. [15], where it was observed that to this order in perturbation theory the equivalent Hermitian Hamiltonian h represents a quartic anharmonic oscillator having a position-dependent mass. However, this interpretation holds only for the case in which we take the coefficients $a^{(P)}$ to vanish for all P .

For each nonzero value of $a^{(P)}$ the contribution to the commutator in (36) for $Q_{1, \text{homogeneous}}^{(P)}$ in (30) is

$$\begin{aligned} \frac{1}{4}i\epsilon^2 [q^3, Q_{1, \text{homogeneous}}^{(P)}] &= \frac{i\epsilon^2 a^{(P)}}{4\mu^{P+4}} \sum_{k=0}^{\infty} \frac{\Gamma(k - P - \frac{3}{2})}{k! \Gamma(-P - \frac{3}{2})} (-\mu^2)^k [q^3, T_{2P+3-2k, 2k}] \\ &= \frac{\epsilon^2 a^{(P)}}{2\mu^{P+4}} \sum_{k=0}^{\infty} \frac{(-\mu^2)^k}{k!} \\ &\quad \times \left[\frac{(P+1-k)\Gamma(k - P + \frac{1}{2})}{\Gamma(-P - \frac{3}{2})} T_{2P-2k, 2k} - \frac{3k\Gamma(k - P - \frac{3}{2})}{\mu^2 \Gamma(-P - \frac{3}{2})} T_{2P+4-2k, 2k} \right], \end{aligned} \quad (39)$$

where we have made use of (37).

We have shown in this paper that the \mathcal{C} operator contains an infinite number of arbitrary parameters. It remains an open question as to whether there is an additional physical or mathematical condition that would determine the \mathcal{C} operator uniquely; that is, whether there is an advantageous or a “best” choice for the arbitrary parameters $a^{(P)}$ in (31). For example, one might try to choose $a^{(P)}$ such that the equivalent Hermitian Hamiltonian h has the form $p^2 + V(q)$. [The non-Dirac-Hermitian \mathcal{PT} -symmetric Hamiltonian $H = p^2 - gq^4$ is spectrally equivalent to the Dirac-Hermitian Hamiltonian $h = p^2 + 4gq^4 - 2\sqrt{g}\hbar q$ [16, 17, 18, 19, 20].] However, there appears to be no choice of the parameters $a^{(P)}$ that achieves such a simple

form. Thus, we conclude by posing the question, is there a useful fourth constraint that one can use to supplement the three conditions in (2), (3), and (4), that gives rise to the “best form” for the \mathcal{C} operator?

We conjecture that the answer to this question in the context of quantum field theory is that the huge parametric freedom in the \mathcal{C} operator can be used to impose the condition of *locality*. A \mathcal{C} operator for a $g\varphi^3$ quantum field theory has been calculated perturbatively to first order in g [4]. The resulting \mathcal{Q} operator was found to decay exponentially at large spatial distances because it contains an associated Bessel function. It may be possible to exploit the parametric freedom in \mathcal{C} to replace the Bessel function by a spatial delta function so that the \mathcal{C} operator becomes local.

Acknowledgments

CMB is grateful to the Graduate School of the University of Heidelberg, where this work was done, for its hospitality. CMB thanks the U.S. Department of Energy for financial support.

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- [1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
 - [2] P. Dorey, C. Dunning and R. Tateo, J. Phys. A **34** L391 (2001); *ibid.* **34**, 5679 (2001).
 - [3] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002) and Am. J. Phys. **71**, 1095 (2003).
 - [4] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **93**, 251601 (2004) and Phys. Rev. D **70**, 025001 (2004).
 - [5] See, for example, G. Levai and M. Znojil, J. Phys. A: Math. Gen. **33**, 7165 (2000).
 - [6] C. M. Bender, P. N. Meisinger, and Q. Wang, J. Phys. A: Math. Gen. **36**, 1973 (2003).
 - [7] C. M. Bender and B. Tan, J. Phys. A: Math. Gen. **39**, 1945 (2006).
 - [8] K. A. Milton, Czech. J. Phys. **54**, 85 (2004); C. M. Bender, I. Cavero-Pelaez, K. A. Milton, and K. V. Shajesh, Phys. Lett. B **613**, 97 (2005).
 - [9] C. M. Bender and H. F. Jones, Phys. Lett. A **328**, 102 (2004); C. M. Bender, J. Brod, A. Refig, and M. E. Reuter, J. Phys. A: Math. Gen. **37**, 10139 (2004); C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, Phys. Rev. D **71**, 025014 (2005); C. M. Bender, H. F. Jones and R. J. Rivers, Phys. Lett. B **625**, 333 (2005); C. M. Bender, S. F. Brandt, J.-H. Chen, and Q. Wang, Phys. Rev. D **71**, 065010 (2005).
 - [10] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002); J. Phys. A: Math. Gen. **36**, 7081 (2003).
 - [11] F. Scholtz, H. Geyer, and F. Hahne, Ann. Phys. **213**, 74 (1992).
 - [12] C. M. Bender and G. V. Dunne, Phys. Rev. D **40**, 10 (1989).
 - [13] C. M. Bender, L. R. Mead, and S. S. Pinsky, J. Math. Phys. **28**, 509 (1987).
 - [14] I. Mendaš and P. Milutinović, J. Phys. A: Math. Gen. **22**, L687 (1989).
 - [15] A. Mostafazadeh, J. Phys. A: Math. Gen. **38**, 6557 (2005); Erratum, *ibid.* **38**, 8185 (2005).
 - [16] A. A. Andrianov, Ann. Phys. **140**, 82 (1982).
 - [17] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. **26**, 5541 (1993).
 - [18] H. F. Jones and J. Mateo, Phys. Rev. D **73**, 085002 (2006).
 - [19] C. M. Bender, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie, Phys. Rev. D **74**, 025016 (2006).

[20] A. A. Andrianov, Phys. Rev. D **76**, 025003 (2007).